Crystal Graph and Littlewood Richardson rule(Combinatorial Aspects in Representation Theory and Geometry)

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§0. Introduction

Recently, Professor Kashiwara constructed "Crystal base". We shall introduce the crystal base and the crystal graph associated with it, and their applications.

§1. Crystal base

1.1 Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with the Cartan subalgebra $\mathfrak{t}$ the set of simple roots $\{\alpha_i \in \mathfrak{t}^*\}$ and the set of simple coroots $\{h_i \in \mathfrak{t}\}$. We take an inner product $(\quad,\quad)$ on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in \mathbf{Z}_{>0}$ and $< h_i, \lambda >= \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$. Then the q-analogue $U_q(\mathfrak{g})$ is the algebra over $\mathbf{Q}(q)$ generated by e_i, f_i and the invertible elements t_i satisfying the following relations;

(1.1.1)
$$t_i t_j = t_j t_i$$

(1.1.2)
$$t_i e_j t_i^{-1} = q^{2(\alpha_j,\alpha_j)} e_j$$
$$t_i f_j t_i^{-1} = q^{-2(\alpha_j,\alpha_j)} f_j$$

(1.1.3)
$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \text{where} \quad q_i = q^{(\alpha_i,\alpha_i)}$$

For $i \neq j$, we have, setting $b = 1- < h_i, \alpha_j >$

(1.1.4)
$$\sum_{\mu=0}^{b} e_i^{(\mu)} e_j e_i^{(b-\mu)} = 0$$
$$\sum_{\mu=0}^{b} f_i^{(\mu)} f_j f_i^{(b-\mu)} = 0$$

Here $e_i^{(k)} = \frac{e_i^k}{[k]_i!}$ $f_i^{(k)} = \frac{f_i^k}{[k]_i!}$, $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $[k]_i! = \prod_{n=1}^{k} [n]_i$.

For a finite dimensional $U_q(\mathfrak{g})$-module M, we set for $\lambda \in P = \{\lambda \in \mathfrak{t}^*; < h_i, \lambda > \in \mathbf{Z}\}$ $M_\lambda = \{u \in M; t_i u = q^{2(\alpha_i,\lambda)}u\}$. We call M integrable if $M = \oplus M_\lambda$. Then we have

$$M_\lambda = \bigoplus_{0 \leq k \leq < h_i,\lambda >} f_i^{(k)}(M_\lambda \cap \mathrm{Ker}\, e_i) \tag{1.1.5}$$

We define the operators $\tilde{e}_i, \tilde{f}_i$ acting on M by

$$\tilde{e}_i f_i^{(k)} u = f_i^{(k-1)} u \quad and \quad \tilde{f}_i f_i^{(k)} u = f_i^{(k+1)} u \tag{1.1.6}$$

for $u \in M_\lambda \cap \mathrm{Ker}\, e_i$ and (λ, k) as above.

Definition 1.1.1. *A pair (L, M) is called a crystal base of a finite-dimensional integrable representation M if the following condition are satisfied:*

(1.1.7) *L is a free sub-A-module of M such that $\mathbf{Q}(q) \otimes_A L \cong M$.*

Here A is the ring of rational functions regular at $q = 0$

(1.1.8) *B is a base of the $\mathbf{Q}$-vector space L/qL*

$$L = \oplus L_\lambda, \quad B = \coprod B_\lambda \tag{1.1.9}$$

where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda / qL_\lambda)$

$$\tilde{f}_i L \subset L, \quad \tilde{e}_i L \subset L \tag{1.1.10}$$

$$\tilde{f}_i B \subset B \cup \{0\} \quad and \quad \tilde{e}_i B \subset B \cup \{0\} \tag{1.1.11}$$

(1.1.12) *For u, $v \in B$ and $i \in I$, $u = \tilde{e}_i v$ if and only if $v = \tilde{f}_i u$.*

Then the following results are proved in [K] when $\mathfrak{g} = A_n, B_n, C_n$ and D_n and announce in [K] in general case. Let $\lambda \in P_+ = \{\lambda \in \mathfrak{t}^*; < h_i, \lambda > \in \mathbf{Z}_{\geq 0}\}$ and $V(\lambda)$ be the irreducible integrable $U_q(\mathfrak{g})$-module generated by the highest weight vector u_λ of weight λ. Let $L(\lambda)$ be the sub A-module generated by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda$ and let $B(\lambda)$ be the subset of the $L(\lambda)/qL(\lambda)$ consisting of the non-zero vector of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda \bmod qL(\lambda)$.

Theorem 1.1.2. *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

Theorem 1.1.3. *Let $M \in \mathcal{O}_{int}$ and (L, B) is a crystal base of M. Then there is an isomorphism*

$$M \cong \oplus_j V(\lambda_j) \text{ by which } (L, B) \cong \oplus_j (L(\lambda_j), B(\lambda_j)).$$

Theorem 1.1.4. *Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1, 2)$. Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2;\ b_j \in B_j\ (j = 1, 2)\} \subset L/qL$. Then we have*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if there exists } n \geq 1 \text{ such that } f_i^n b_1 \neq 0 \text{ and } e_i^n b_2 = 0. \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if there exists } n \geq 1 \text{ such that } e_i^n b_1 \neq 0 \text{ and } f_i^n b_2 = 0. \\ \tilde{e}_i b_1 \otimes b_2 & \text{otherwise.} \end{cases}$$

Defintion 1.1.5. *A crystal graph of a crystal base (L, B) is the colored oriented graph B, by the rule:*

$$u \xrightarrow{i} v \qquad \Longleftrightarrow \qquad v = \tilde{f}_i u.$$

Example 1.1.6

Let V_i is the $i + 1$-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$.

(i) For $\mathfrak{g} = \mathfrak{sl}_2$, the crystal graph of V_2 is given by as follows;

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

(ii) For $\mathfrak{g} = \mathfrak{sl}_2$, the crystal graph of V_3 is given by as follows;

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(iii) By Theorem 1.1.4, the crystal graph of $V_3 \otimes V_2$ is described by as follows;

$$\begin{array}{ccccccccc}
 & V_3 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
V_2 & & & & & & & & \\
\bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\downarrow & & & & & & & & \downarrow \\
\bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet \\
\downarrow & & & & & & \downarrow & & \downarrow \\
\bullet & & \bullet & \longrightarrow & \bullet & & \bullet & & \bullet
\end{array}$$

Here, we get that the crystal graph of $V_3 \otimes V_2$ are decomposed into three connected components. They corrspond to V_5, V_3 and V_1 respectively.

§2. Remarks on the crystal graphs

2.1 Let us investigate first the crystal graph of the tensor product of the vector representation of $U_q(sl_2)$. The crystal graph of the vector representation is $u_+ \longrightarrow u_-$. The crystal graph of the trivial representation is u_0. We shall calculate $\tilde{e}(u_{i_1} \otimes \cdots \otimes u_{i_N})$ and $\tilde{f}(u_{i_1} \otimes \cdots \otimes u_{i_N})$, where e and f are generators of $U_q(sl_2)$, and $i_1, \cdots, i_N = +, -, 0$.

Proposition 2.1.1. *For* $u = u_{i_1} \otimes \cdots \otimes u_{i_N}$ $(i_j = +, -, 0)$, *the actions of* $\tilde{e}$ *and* $\tilde{f}$ *are given by the following three steps;*

(I) We neglect u_0

(II) If there is $u_+ \otimes u_-$ in u, then we neglect such a pair.

(III) Then $\tilde{e}$ changes the u_- in the most right to u_+ and $\tilde{f}$ changes the u_+ in the most left to u_-. If there is no u_- (resp. u_+), then $\tilde{e}u = 0$ (resp. $\tilde{f}u = 0$).

Example For $u = u_- \otimes u_0 \otimes \underbrace{u_+ \otimes \overbrace{u_+ \otimes u_-} \otimes u_-} \otimes u_+$,

$$\tilde{e}u = u_+ \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_+$$

$$\tilde{f}u = u_- \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_-$$

2.2 Now let $U_q(\mathfrak{g})$ be the q-analogue as in §1. and let $\lambda_1, \cdots, \lambda_N \in P_+$ and $\lambda = \sum_i \lambda_i$. Then there is a unique embedding $V(\lambda) \hookrightarrow V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$ sending u_λ to $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_N}$. Hence $B(\lambda)$ is embedded into $\otimes_{j=1}^N B(\lambda_j)$.

Proposition 2.2.1. *Assume the following condition for any* k $(1 \leq k < N)$.

(2.2.1) *If* $u \in B(\lambda_{k+1})$ *satisfies*

(i) $u_{\lambda_k} \otimes u \in B(\lambda_k + \lambda_{k+1})$

(ii) $\tilde{e}u = 0$ *for any* i *such that* $< h_i, \lambda_\nu >= 0$ *for* $\nu \leq k$,

then $u = u_{\lambda_{k+1}}$

Then we have

$$(2.2.2)\qquad V(\lambda)\cong\bigcap_{k=1}^{N-1}V(\lambda_1)\otimes\cdots\otimes V(\lambda_{k-1})\otimes V(\lambda_k+\lambda_{k+1})\otimes V(\lambda_{k+2})\otimes\cdots B(\lambda_N),$$

$$(2.2.3)\qquad B(\lambda)\cong\bigcap_{k=1}^{N-1}B(\lambda_1)\otimes\cdots\otimes B(\lambda_{k-1})\otimes B(\lambda_k+\lambda_{k+1})\otimes B(\lambda_{k+2})\otimes\cdots B(\lambda_N).$$

§3. Crystal Graphs for $U_q(C_n)$-modules

3.1 Notation

In the rest of this paper we shall treat the C_n-case. Let $(\varepsilon_1,\cdots,\varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of C_n such that $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ $(1\le i<n)$ and $\alpha_n=2\varepsilon_n$ form the set of simple roots. Hence, α_n is the long roots and $\alpha_1,\cdots,\alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1\le i\le n}$ be the dual base of $\{h_i\}_{1\le i\le n}$. Hence $\Lambda_i=\varepsilon_1+\cdots+\varepsilon_i$ $(1\le i\le n)$.

3.2 The crystal graph of the vector representation.

First let us consider the vector representation $V(\Lambda_1)=V_\Box$. Letting $\boxed{i}$, $\boxed{\bar{i}}$ $(1\le i\le n)$ be the base of $\mathbf{Q}(q)^{\oplus 2n}$, the vector representation of $U_q(C_n)$ is explicitly constructed as follows;

$$(3.2.1)\qquad \begin{aligned} t_i\,\boxed{j}&=q^{2(h_i,\varepsilon_j)}\boxed{j} &,\ t_i\,\boxed{\bar{j}}&=q^{2(h_i,-\varepsilon_j)}\boxed{\bar{j}}\\ e_i\,\boxed{j}&=\delta_{i+1,j}\boxed{j-1} &,\ e_i\,\boxed{\bar{j}}&=\delta_{i,j}\boxed{\overline{j+1}}. \qquad (1\le i<n,\ 1\le j\le n)\\ f_i\,\boxed{j}&=\delta_{i,j}\boxed{j+1} &,\ f_i\,\boxed{\bar{j}}&=\delta_{i+1,j}\boxed{\overline{j-1}}\end{aligned}$$

and

$$(3.2.2)\qquad \begin{aligned} e_n\,\boxed{j}&=0,\quad e_n\,\boxed{\bar{j}}=\delta_{j,n}\boxed{n}\\ f_n\,\boxed{j}&=\delta_{j,n}\boxed{\bar{n}} \quad ,\ f_n\,\boxed{\bar{j}}=0\end{aligned}\qquad (1\le j\le n)$$

Here, we understand $\boxed{i}=\boxed{\bar{i}}=0$ unless $1\le j\le n$. Then the crystal base $(L(V_\Box),B(V_\Box))$ is ginen by

$$(3.2.3)\qquad \begin{aligned} L(V_\Box)&=\bigoplus_{i=1}^{n}(A\,\boxed{i}\oplus A\,\boxed{\bar{i}}\,)\\ B(V_\Box)&=\{\,\boxed{i}\,,\,\boxed{\bar{i}}\ ;\ 1\le i\le n\},\end{aligned}$$

and the crystal graph of $V_\Box$ is given by;

(3.2.4) $$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Remark that we have

(3.2.5) $$\tilde{e}_i^2 = \tilde{f}_i^2 = 0 \quad \text{on } B(V_\Box)$$

Hence, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(V_\Box)^{\otimes m}$ is given by Proposition 2.1.1.

3.3 The crystal graph of the fundamental representations

The representation $V(\Lambda_N)$ with highest weighgt Λ_N $(1 \le N \le n)$ is embedded into $V_\Box^{\otimes N}$. Similarly to the A_n-case, the connected component of the crystal graph of $B(V_\Box)^{\otimes N}$ containing $\boxed{1} \otimes \cdots \otimes \boxed{N}$ is that of $B(\Lambda_N)$.

We write $\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array}$ for $\boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$

We denote by u_{Λ_N} the highest weight vector $\boxed{1} \otimes \cdots \otimes \boxed{N}$. We give the linear order on $\{i,\ \bar{i}\ ;\ 1 \le i \le n\}$ by

(3.3.1) $$1 \prec 2 \prec \cdots n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}.$$

This ordering is derived by the crystal graph (3.2.4) of $V_\Box$. We set

(3.3.2) $$I_N^{(C)} = \left\{ \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_\Box)^{\otimes N}\ ;\ \begin{array}{l} (1)\ \ 1 \preceq i_1 \prec \cdots i_N \preceq \bar{1}, \\ (2)\ \ \text{if } i_k = p \text{ and } i_l = \bar{p}, \\ \qquad \text{then } k + (N - l + 1) \le p \end{array} \right\}.$$

Proposition 3.3.2. *$B(\Lambda_N)$ coincides with $I_N^{(C)}$*

Remark 3.3.3.

(i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ *for* $1 \le i \le n$ *and* $\tilde{e}_n^2 = \tilde{f}_n^2 = 0$ *on* $B(\Lambda_N)$.

(ii) *If* $u \in B(\Lambda_N)$ *satisfies* $\tilde{f}_i^2 u \ne 0$, *then* u *contains* i *and* $\overline{i+1}$ *but neither* $i+1$ *nor* $\bar{i}$. *If* $u \in B(\Lambda_N)$ *satisfies* $\tilde{e}_i^2 u \ne 0$, *then* u *contains* $i+1$ *and* $\bar{i}$ *but neither* i *nor* $\overline{i+1}$.

(iii) *If* $u \in B(\Lambda_N)$ *satisfies* $\tilde{f}_i u \ne 0$ *and* $\tilde{e}_i u \ne 0$, *then* u *contains* $i+1$, $\overline{i+1}$ *but neither* i *nor* $\bar{i}$.

3.4 The crystal graph of $V(\Lambda_M + \Lambda_N)$

Now, we shall investigate the crystal graph of $V(\Lambda_M+\Lambda_N)$ with $1 \le M \le N \le n$. By embedding $V(\Lambda_M+\Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M+\Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containning $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

$$\text{For} \quad u = \begin{array}{|c|}\hline j_1 \\ \vdots \\ j_M \\ \hline\end{array} \in B(\Lambda_M) \quad \text{and} \quad v = \begin{array}{|c|}\hline i_1 \\ \vdots \\ \\ i_N \\ \hline\end{array} \in B(\Lambda_N),$$

the vector $u \otimes v \in B(\Lambda_M) \otimes B(\Lambda_N)$ will be denoted by $\begin{array}{|c|c|}\hline i_1 & j_1 \\ \vdots & \vdots \\ & j_M \\ \cline{2-2} i_N & \\ \cline{1-1}\end{array}$

Definition 3.3.1. *For* $1 \le i \le j \le n$, *we say that* $w \in B(\Lambda_M) \otimes B(\Lambda_N)$ *is in* (i,j)-*configuration if* w *holds the following; (3.4.1) There exist* $1 \le p \le q << \le r \le s$ *such that* $i_p = i$, $j_q = j$, $j_r = \bar{j}$, $j_s = \bar{i}$ *or* $i_p = i$, $i_q = j$, $i_r = \bar{j}$, $j_s = \bar{i}$. *Remark that when* $i = j$, *we understand that* $p = q$ *and* $r = s$. *Now, we define* $p(i,j;w) = (q-p)+(s-r)$, *remark that if there exist another* $1 \le p' \le q' < r' \le s'$ *such that gives* (i,j)-*configuration on* w, *we take the largest one as* $p(i,j;w)$. *Let us set*

$$(3.4.2) \quad I^{(C)}_{(M,N)} = \left\{ w = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \vdots & \vdots \\ \vdots & j_M \\ \cline{2-2} i_N \\ \cline{1-1} \end{array} \in B(\Lambda_M) \otimes B(\Lambda_N); \quad \begin{array}{c} w \text{ satisfies the conditions} \\ \text{(M.N.1) and (M.N.2)} \end{array} \right\}$$

(M.N.1) $\quad i_k \leq j_k$ *for* $1 \leq k \leq M$

(M.N.2) if w is in the (i,j)-configuration, then $p(i,j;w) < j - i$.

Remarmk that any vector of $I^{(C)}_{(M,N)}$ is not in the (i,i)-configuration, because $p(i,j;w) \geq 0$.

Proposition 3.4.3. *$B(\Lambda_M + \Lambda_N)$ coincides with $I^{(C)}_{(M,N)}$.*

3.5 The crystal graph of $V(\lambda)$

Let $\lambda = \sum_{i=1}^{p} \Lambda_{l_i} \quad (1 \leq l_1 \leq l_2 \leq \cdots \leq n)$ be a dominant integral weight. Let us consider the crystal graph of $B(\lambda)$. By Lemma 3.4.4, we can apply Proposition 2.2.1 and hence

$$B(\lambda) = \left\{ u_1 \otimes \cdots \otimes u_p \in B(\Lambda_{l_1}) \otimes \cdots \otimes B(\Lambda_{l_p}); \ u_i \otimes u_{i+1} \in B(\Lambda_{l_i} + \Lambda_{l_{i+1}}) \ \text{ for } \ 1 \leq i \leq p \right\}$$

For the Young diagram Y with the columns of $l_i \ (1 \leq i \leq p)$, we define a C-semi-standard tableau with shape Y with elements $\{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\}$ in each boxes of Y satisfying the following conditions;

(3.5.1) Letting $t_{i,j}$ be the element of $\{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\}$ at the i-th column and j-th row, we have

$$t_{i,j} \leq t_{i+1,j} \qquad \text{and} \qquad t_{i,j} < t_{i,j+1}$$

(3.5.2) For $1 \leq p \leq q \leq n$, if $t_{i,j} = p, \quad t_{i+1,j'} = \overline{p}$ and if $t_{i,k} = q, \quad t_{i,k'} = \overline{q}$ (resp. $t_{i+1,k} = q, \quad t_{i+1,k'} = \overline{q}$) then $(k - j) + (j' - k') < q - p$.

Theorem 3.5.1. *$B(\lambda)$ coincides with the set of the C-semi-standard tableaux with shape Y. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are described by Proposition 2.1.2 by identifying i and $\overline{i+1}$ with u_+, $i+1$ and $\overline{i}$ with u_-, and others with u_0.*

§4. Littlewood Richardson rule for C_n.

In this section, we give the rule to decompose $V_Y \otimes V_{Y'}$ (Y and Y' are Young diagrams with depth n) in terms of crystal graph.

4.1 The following lemma plays a significant role in this rule.

Lemma 4.1.1. *For $u \in B(V_Y)$ and $v \in B(V_{Y'})$,*

$$\tilde{e}_i(u \otimes v) = 0 \ \text{(for any } i) \iff \begin{array}{l} \bullet\ \tilde{e}_i u = 0 \\ \bullet\ \tilde{e}_i^{<h_i,\lambda>+1} v = 0 \ \text{(for any } i) \end{array}$$

where λ is the weight of Y.

4.2. Decomposition of $V_Y \otimes V_\square$

Lemma 4.2.1. *For a Young diagram $Y = (l_1, l_2, \cdots, l_n)$, when we identify Y and the highest element u_Y of $B(V_Y)$, where u_Y is the following;*

$$\begin{array}{l} 111 \cdots\cdots\cdots\cdots 111 \\ 222 \cdots\cdots\cdots 222 \\ \cdots\cdots\cdots \\ ii \cdots\cdots ii \\ \cdots\cdots \\ n \cdots n \end{array} \qquad\qquad \#\{i \in u_Y\} = l_i$$

a) For $\quad \in B(V_\square)$ $(j = 1, 2, \cdots, n)$, $Y \otimes \quad$ is the highest element of $B(V_Y \otimes V_\square)$ if and only if $l_{j-1} - l_j > 0$. b) For $\quad \in B(V_\square)$ $(j = 1, 2, \cdots, n)$, $Y \otimes \quad$ is the highest element of $B(V_Y \otimes V_\square)$ if and only if $l_j - l_{j+1} > 0$

Remark that we assume $l_0 = \infty$ and $l_{n+1} = 0$.

Proof We can easily obtain the result by Lemma 4.1.1 and the followinf facts;

For any i,

$$\tilde{e}_i^{<h_i,\lambda>+1} \boxed{j} = 0 \iff < h_{j-1}, \lambda >= l_{j-1} - l_j > 0$$

$$\tilde{e}_i^{<h_i,\lambda>+1} \boxed{\bar{j}} = 0 \iff < h_j, \lambda >= l_j - l_{j+1} > 0$$

q.e.d.

Now, we get the following proposition.

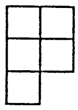
Proposition 4.2.2. *Let $Y = (l_1, l_2, \cdots, l_n)$ be a Young diagram and V_Y be a finite dimensional irreducible C_n-module characterized by Y,*

$$V_Y \otimes V_{\square} \cong \bigoplus_{j=1}^{n} V_{(Y \longleftarrow j)} \oplus \bigoplus_{j=1}^{n} V_{(Y \longleftarrow \bar{j})}$$

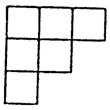
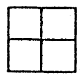
where $(Y \longleftarrow j) = (l_1, \cdots, l_j + 1, \cdots, l_n)$ and $(Y \longleftarrow \bar{j}) = (l_1, \cdots, l_j - 1, \cdots, l_n)$.

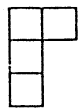
Remark 4.2.3. If Y is not a Young diagram, then V_Y means a 0-dimensional vector space.

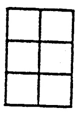
Proof By Lemma 4.1.1 and Lemma 4.2.1 we can identify $u \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) with highest condition with a Young diagram $(Y \longleftarrow j)$ (resp. $(Y \longleftarrow \bar{j})$) Hence, $Y \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) is the highest element of $B(V_Y \otimes V_{\square})$ if and only if $(Y \longleftarrow j)$ (resp. $(Y \longleftarrow \bar{j})$) is a Young diagram. Since both $Y \otimes \boxed{j}$ (resp. $Y \otimes \boxed{\bar{j}}$) and $(Y \longleftarrow j)$ (resp. $(Y \longleftarrow \bar{j})$) have the same weight, $u \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) $(1 \leq j \leq n)$ with the highest condition can be identified with a Young diagram $(Y \longleftarrow j)$ (resp. $(Y \longleftarrow \bar{j})$). *q.e.d.*

Example 4.2.4. For $\mathfrak{g} = C_3$ and $Y = (2,2,1) =$ 

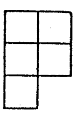
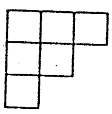
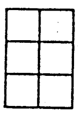
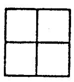
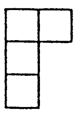
$$B(V_{\square}) = \{ \boxed{1}, \boxed{2}, \boxed{3}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}} \}$$

$Y \otimes \boxed{1} = (Y \longleftarrow 1) = (3,2,1) =$  $Y \otimes \boxed{\bar{3}} = (Y \longleftarrow \bar{3}) = (2,2,0) =$ 

$Y \otimes \boxed{2} = (Y \longleftarrow 2) = (2,3,1) \quad \times$ $Y \otimes \boxed{\bar{2}} = (Y \longleftarrow \bar{2}) = (2,1,1) =$ 

$Y \otimes \boxed{3} = (Y \longleftarrow 3) = (2,2,2) =$  $Y \otimes \boxed{\bar{1}} = (Y \longleftarrow \bar{1}) = (1,2,1) \quad \times$

Then, we get

 $\otimes \ \square =$  $\oplus$  $\oplus$  $\oplus$ 

4.3. Decomposition of $V_Y \otimes V_{Y'}$

We shall treat a general case. Let Y and Y' be Young diagrams. We give a combinatorial description for irreducible decomposition of $V_Y \otimes V_{Y'}$. By the following lemma and the way of the construction of the crystal graph, we know that the previous elementary case plays a significant role in a general case.

Lemma 4.3.1. *Let $J = \{1, 2, \cdots, p\}$ be a finite index set and V_j $(j \in J)$ be a finite dimensional irreducible representation of $U_q(C_n)$. For $u_1 \otimes u_2 \otimes \cdots \otimes u_p \in B(\bigotimes_{j \in J} V_j)$, following two assertions are equivalent;*

(A) $u_1 \otimes u_2 \otimes \cdots \otimes u_p$ *is the highest element of* $B(\bigotimes_{j \in J} V_j)$

(B) *For any* $j \in J$, $u_1 \otimes u_2 \otimes \cdots \otimes u_p$ *is the highest element of* $B(V_1 \otimes \cdots \otimes V_j)$

Proof First assuming (B), we get (A) easily. Next we assume (A). For any $j \in J$ we can consider $u_1 \otimes u_2 \otimes \cdots \otimes u_p = (u_1 \otimes \cdots \otimes u_j \otimes u_{j+1} \otimes \cdots \otimes u_p) \in B(V_1 \otimes \cdots \otimes V_j) \otimes B(V_{j+1} \otimes \cdots V_p)$. By Lemma 4.1.1, if $(u_1 \otimes \cdots \otimes u_j \otimes u_{j+1} \otimes \cdots \otimes u_p)$ satisfies the highest condition, $u_1 \otimes \cdots \otimes u_j$ also satisfies the highest condition. Hence, we get (B). *q.e.d.*

Here, by Proposition 4.2.2 and Lemma 4.3.1, we obtain the following theorem.

Theorem 4.3.2. *Let Y and Y' be Young diagrams. Let m be $\#Y'$. Then we obtain the following;*

$$V_Y \otimes V_{Y'} \cong \bigoplus_{\boxed{j_1} \otimes \cdots \otimes \boxed{j_m} \in B(V_{Y'})} V_{((((Y \longleftarrow j_1) \longleftarrow j_2) \cdots) \cdots \longleftarrow j_m)}$$

where $V_{((((Y \longleftarrow j_1) \longleftarrow j_2) \cdots) \cdots \longleftarrow j_m)}$ is a 0-dimensional vector space if there exists $k \in \{1, \cdots, m\}$ such that $((((Y \longleftarrow j_1) \longleftarrow j_2) \cdots) \cdots \longleftarrow j_k)$ is not a Young diagram. ∎

Example 4.3.3. *For* $\mathfrak{g} = C_2$, $Y = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} = (2,2)$ *and* $Y' = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = (1,1)$.

$$B(V_{Y'}) = \left\{ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array}, \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline\end{array} \right\}$$

$$Y \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} = ((\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \longleftarrow 1) \longleftarrow 2) = (\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array} \longleftarrow 2) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline\end{array}$$

$$Y \otimes \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline\end{array} = ((\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \longleftarrow 1) \longleftarrow \bar{2}) = (\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array} \longleftarrow \bar{2}) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$$

$$Y \otimes \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline\end{array} = ((\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \longleftarrow 2) \longleftarrow \bar{2}) = (\begin{array}{|c|c|c|}\cline{1-2} \ & \ \\ \hline \ & \ & \ \\ \hline\end{array} \longleftarrow \bar{2}) \qquad \times$$

$$Y \otimes \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array} = ((\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \longleftarrow 2) \longleftarrow \bar{1}) = (\begin{array}{|c|c|c|}\cline{1-2} \ & \ \\ \hline \ & \ & \ \\ \hline\end{array} \longleftarrow \bar{1}) \qquad \times$$

$$Y \otimes \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline\end{array} = ((\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \longleftarrow \bar{2}) \longleftarrow \bar{1}) = (\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array} \longleftarrow \bar{1}) = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$$

Then we get (we omit "V")

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$$

Remark We have already obtained the similar conclusions for A_n, B_n and D_n.

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